FIELD OF FRACTIONS

Every integral domain can be embedded in a field (see proof below). That is, using concepts from set theory, given an arbitrary integral domain (such as the integers), one can construct a field that contains a subset isomorphic to the integral domain. Such a field is called the field of fractions of the given integral domain.

1. Examples

- The rational numbers ($\mathbb{Q}$) is the field of fractions of the integers ($\mathbb{Z}$);
- The Gaussian rational numbers ($\mathbb{Q}[i]$) is the field of fractions of the integers ($\mathbb{Z}[i]$)

Theorem. *Every integral domain can be embedded in a field.*

Proof. Let R be an integral domain. That is, a commutative ring with unity in which the zero-product rule holds. Now consider the set $\bar{F} = R \times R^*$, the set of all ordered pairs of elements in R, excluding those in which the second element is 0.

We will now define a binary relation $\cong$ on $\bar{F}$, which we claim to be an equivalence relation by the following criteria: $(p, q) \cong (\hat{p}, \hat{q})$, if and only if $p \cdot \hat{q} = \hat{p} \cdot q$, for all $(p, q), (\hat{p}, \hat{q}) \in \bar{F}$.

To show that $\cong$ is an equivalence relation, we must show that the reflexive, symmetric, and transitive properties hold. To that end, let $(p, q), (\hat{p}, \hat{q}), (\tilde{p}, \tilde{q}) \in \bar{F}$.

Reflexive

Since $p \cdot q = p \cdot q$, $(p, q) \cong (p, q)$.

Symmetric

Suppose that $(p, q) \cong (\hat{p}, \hat{q})$. Then $p \cdot \hat{q} = \hat{p} \cdot q$, and so by the symmetric property of equality, $\hat{p} \cdot q = p \cdot \hat{q}$. Thus, $(\hat{p}, \hat{q}) \cong (p, q)$.

Transitive

Now suppose that $(p, q) \cong (\hat{p}, \hat{q})$ and $(\hat{p}, \hat{q}) \cong (\tilde{p}, \tilde{q})$. Then $p \cdot \hat{q} = \hat{p} \cdot q$ and $\hat{p} \cdot \tilde{q} = \tilde{p} \cdot \hat{q}$. Multiplying these equations together, we obtain $p\hat{q}\hat{p}\tilde{q} = \hat{p}q\tilde{p}\hat{q}$. Thus, $p\hat{q}\hat{p}\tilde{q} - \hat{p}q\tilde{p}\hat{q} = 0$, and so $\hat{p}\hat{q}(p\tilde{q} - q\tilde{p}) = 0$. Since $\hat{q} \neq 0$, we must have $\hat{p}(p\tilde{q} - q\tilde{p}) = 0$, as R has the zero-product rule.

In the case that $\hat{p} \neq 0$, we will have $p\tilde{q} - q\tilde{p} = 0$, which implies $p\tilde{q} = \tilde{p}q$ and that $(p, q) \cong (\tilde{p}, \tilde{q})$.

Otherwise, if $\hat{p} = 0$, then $p \cdot \hat{q} = 0 \cdot q = 0$, which then implies that $p = 0$. Similarly, we will also find that $\tilde{p} = 0$. This is the special case that $p\tilde{q} = 0 = \tilde{p}q$, and so $(p, q) \cong (\tilde{p}, \tilde{q})$.

Thus, $\cong$ is an equivalence relation, and so we will now define $F = \bar{F}/\cong$, the set of all equivalence classes, and use the notation $\frac{p}{q}$ to denote the element $[p, q]$, the equivalence class containing (p, q).

We now must determine addition and multiplication operations on F and show that F is a field. We claim that addition and multiplication can be given by:

- $\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$, for all $\frac{p}{q},\frac{r}{s}\in F$;
- $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$, for all $\frac{p}{q},\frac{r}{s}\in F$;

Since $\frac{p}{q}$ and $\frac{r}{s}$ may not be unique representations of elements in F, it is necessary to show that the above rules for addition and multiplication provide for well-defined operations.

To that end, let $\frac{p}{q}$, $\frac{\hat{p}}{\hat{q}}$, $\frac{r}{s}$, and $\frac{\hat{r}}{\hat{s}}$ be elements of F, and suppose that $\frac{p}{q}=\frac{\hat{p}}{\hat{q}}$ and $\frac{r}{s}=\frac{\hat{r}}{\hat{s}}$. We will now show that $\frac{ps+qr}{qs}=\frac{\hat{p}\hat{s}+\hat{q}\hat{r}}{\hat{q}\hat{s}}$ and $\frac{pr}{qs}=\frac{\hat{p}\hat{r}}{\hat{q}\hat{s}}$.

Since $\frac{p}{q}=\frac{\hat{p}}{\hat{q}}$ and $\frac{r}{s}=\frac{\hat{r}}{\hat{s}}$, we have $p\hat{q}=\hat{p}q$ and $r\hat{s}=\hat{r}s$. Then:

$$\begin{aligned}(ps+qr)\,\hat{q}\hat{s} &= ps\hat{q}\hat{s}+qr\hat{q}\hat{s}\\ &= p\hat{q}s\hat{s}+q\hat{q}r\hat{s}\\ &= \hat{p}qs\hat{s}+q\hat{q}\hat{r}s\\ &= (\hat{p}\hat{s}+\hat{q}\hat{r})\,qs\end{aligned}$$

and

$$\begin{aligned}pr\hat{q}\hat{s} &= p\hat{q}r\hat{s}\\ &= \hat{p}q\hat{r}s\\ &= \hat{p}\hat{r}qs\end{aligned}$$

Therfore, $\frac{ps+qr}{qs}=\frac{\hat{p}\hat{s}+\hat{q}\hat{r}}{\hat{q}\hat{s}}$ and $\frac{pr}{qs}=\frac{\hat{p}\hat{r}}{\hat{q}\hat{s}}$.

Now we must show that F is a field. Take $\frac{p}{q}$ and $\frac{r}{s}$ as before, and 0 and 1 as the additive and multiplicative identities in R. Also take $\frac{u}{v}\in F$. It should be noted that for any $n\in R^*$, since $(pn)\,q=p\,(qn)$, $\frac{pn}{qn}=\frac{p}{q}$.

Additive identity

We claim that $\frac{0}{1}$ is the additive identity in F. Observe that $\frac{p}{q}+\frac{0}{1}=\frac{p\cdot 1+q\cdot 0}{q\cdot 1}=\frac{p}{q}$.

Additive inverses

We claim that $\frac{-p}{q}$ is the additive inverse of $\frac{p}{q}$. Observe that $\frac{p}{q}+\frac{-p}{q}=\frac{pq+(-pq)}{qq}=\frac{0}{q^2}=\frac{0q^2}{1q^2}=\frac{0}{1}$.

Commutativity of addition

Observe that $\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}=\frac{rq+sp}{sq}=\frac{r}{s}+\frac{p}{q}$.

Associativity of addition

Observe that $\frac{p}{q}+\left(\frac{r}{s}+\frac{u}{v}\right)=\frac{p}{q}+\frac{rv+su}{sv}=\frac{p(sv)+q(rv+su)}{q(sv)}=\frac{psv+qrv+qsu}{qsv}=\frac{(ps+qr)v+(qs)u}{(qs)v}=\frac{ps+qr}{qs}+\frac{u}{v}=\left(\frac{p}{q}+\frac{r}{s}\right)+\frac{u}{v}$.

Multiplicative identity

We claim that $\frac{1}{1}$ is the multiplicative identity in F. Observe that $\frac{p}{q}\cdot\frac{1}{1}=\frac{p\cdot 1}{q\cdot 1}=\frac{p}{q}$.

Multiplicative inverses

On the supposition that $\frac{p}{q} \neq \frac{0}{1}$, we find that $p \neq 0$. We then claim that $\frac{q}{p}$ is the multiplicative inverse of $\frac{p}{q}$. Observe that $\frac{p}{q} \cdot \frac{q}{p} = \frac{pq}{qp} = \frac{1pq}{1pq} = \frac{1}{1}$.

Commutativity of multiplication

Observe that $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} = \frac{rp}{sq} = \frac{r}{s} \cdot \frac{p}{q}$.

Associativity of multiplication

Observe that $\frac{p}{q} \cdot \left(\frac{r}{s} \cdot \frac{u}{v}\right) = \frac{p}{q} \cdot \frac{ru}{sv} = \frac{p(ru)}{q(sv)} = \frac{(pr)u}{(qs)v} = \frac{pr}{qs} \cdot \frac{u}{v} = \left(\frac{p}{q} \cdot \frac{r}{s}\right) \cdot \frac{u}{v}$.

Distributive property

Observe that $\frac{p}{q} \cdot \left(\frac{r}{s} + \frac{u}{v}\right) = \frac{p}{q} \cdot \frac{rv+su}{sv} = \frac{prv+psu}{q(sv)} = \frac{(prv+psu)q}{q(sv)q} = \frac{(pr)(qv)+(qs)(pu)}{(qs)(qv)} = \frac{pr}{qs} + \frac{pu}{qv} = \frac{p}{q} \cdot \frac{r}{s} + \frac{p}{q} \cdot \frac{u}{v}$.

Now we are left to identify a subset of F that is ring-isomorphic to R. We claim that the subset is $\hat{R} = \left\{\frac{r}{1} : r \in R\right\}$ and the isomorphism is mapping $\phi : R \to \hat{R}$, defined by $\phi(r) = \frac{r}{1}$, or all $r \in R$.

One-to-one

Let $r, s \in R$ and suppose that $\phi(r) = \phi(s)$. Then $\frac{r}{1} = \frac{s}{1}$, and so $r \cdot 1 = 1 \cdot s$. Thus $r = s$.

Onto

Let $\frac{r}{1} \in \hat{R}$. Then $\phi(r) = \frac{r}{1}$.

Preservation of structure

Let $r, s \in R$. Then $\phi(r+s) = \frac{r+s}{1} = \frac{r \cdot 1 + 1 \cdot s}{1 \cdot 1} = \frac{r}{1} + \frac{s}{1} = \phi(r) + \phi(s)$, and $\phi(rs) = \frac{rs}{1} = \frac{r \cdot s}{1 \cdot 1} = \frac{r}{1} \cdot \frac{s}{1} = \phi(r)\phi(s)$. $\square$